Robert Luther

Aus dem Gebiet der Farbreizmetrik (On color stimulus metrics)

Zeitschrift für technische Physik 8 (1927) 540-558

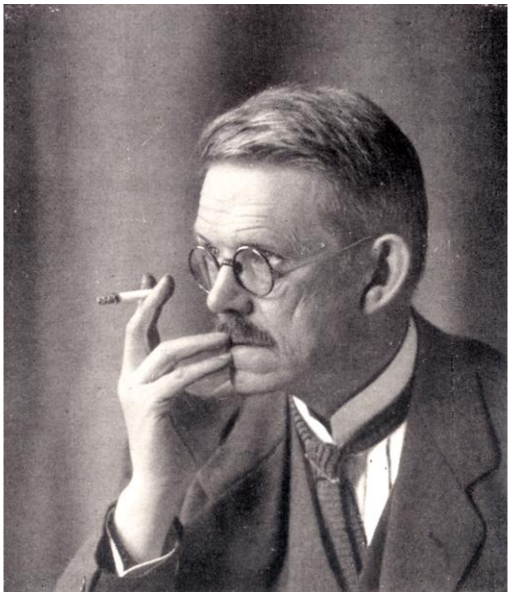
An English translation with a short biographical introduction by Rolf G. Kuehni and a technical introduction by Michael H. Brill

Copyright statement:
Copyright of original paper: unknown; copyright of translation and biographical introduction: Rolf G. Kuehni, 2009; copyright of technical introduction: Michael H. Brill, 2010

Note

Explanatory wording by the translator in angular parentheses. There are three extended explanatory notes by the translator identified with alphabetical superscripts.

Robert Luther 1868-1945



Dresden, ca. 1925 - U. Richter

Robert (Thomas Diedrich) Luther was born in Moscow to German parents on Jan. 2, 1868. His father Alexander was a lawyer. Among his direct ancestors was Hans Luther (1492-1558), a cousin of the reformer Martin Luther. From 1885 to 1889 he studied chemistry at the University of Dorpat in Russia. Toward the end of 1889 he was named assistant to the chemist F. K. Beilstein at the University of St. Petersburg. A serious illness in 1891 forced him to recuperate during the next two years. In 1894 he resumed studying chemistry, this time at the University of Leipzig where he received his PhD degree in 1885. His primary educator there was Wilhelm Ostwald (1853-1931). In 1896 Luther was named an assistant to Ostwald at the Physico-Chemical Institute of the University of Leipzig. In 1899 Luther submitted his habilitation thesis, titled “Equilibrium change between halogen compounds of silver and the free halogens caused by light,” and obtained lecturer status. In the same year he published the monograph “The chemical processes of photography,” a record of six public lectures he gave on the subject. In these lectures he demonstrated, among many other things, the chemical reaction kinetics in layers or volumes of substances, a subject that occupied him for the rest of his life. He also was the co-author of the 1902 second edition of Ostwald’s book on experimental methodology in physico-chemical measurements. In the year 1900 he was named assistant director of the Ostwald Institute at the University of Leipzig.

As colleagues, Ostwald and Luther were considerably different types. Ostwald did not like to have to lecture but published many papers and books. As a result, Luther was burdened with much of the lecturing activities at Ostwald’s institute. Despite important work in many fields he rarely wrote articles. In 1904 Luther was named a regular professor of physical chemistry at the University of Leipzig. Ostwald, wanting to just manage the Institute, was found to be neglecting his lecturing duties and resigned from his position in 1906, retreating to his country estate in Grossbothen, where he launched into his color research. In the same year Luther was named director of the newly-formed photochemical department. In the fall-out of Ostwald's departure Luther found himself in a difficult position in Leipzig and in 1908 accepted an appointment at the Photographic Science Institute of the Technical University of Dresden, organized shortly before with the support of the local photographic industry (such as Zeiss). Luther remained there until his change to professor emeritus status in 1936, performing significant research in photographic and general physical chemistry and also concerning himself with the definition of color stimuli and color stimulus measurement. He was much admired by his students, among them Manfred Richter who later developed the DIN color order system. In 1909-1910 the famous American photographer Imogen Cunningham (1883-1976) was a student of Luther, learning the technique of platinum prints. Luther remained in Dresden where he passed away on April 17, 1945, the last day of the Allied bombing runs on that city.

Luther's interest in color phenomena was the natural outcome of his activities related to the chemistry of color photography. He clearly distinguished between what he considered to be objective definition of color stimuli and perceptual color phenomena.
Most of his seminal paper "On color stimulus metrics," offered here in translation, was ready for publication in 1923 under the title "Color and spectrum," as a contribution to a Festschrift for Ostwald in the *Zeitschrift für angewandte Chemie*. However, the rabid inflation of the time prevented publication until 1927, as described in Footnote 4 of the paper. His only other brief (2 pages) publication on the subject of color, from 1942, is concerned with practical application of the moment sum curve, developed in the 1927 paper.

Sources of biographical information:

1. M. Richter, Robert Luther, Die Farbe 3:133-135 (1955).
2. A. Fischer, Luther, Robert, in: W. Pötsch (ed.) Lexikon bedeutender Chemiker, Frankfurt: Verlag Harry Deutsch, 1989.

Technical Introduction

The article translated here is a rare summary of thoughts on color by Robert Luther. The content is quite heterogeneous, because it represents an accumulation of ideas that could not find earlier publication during a period when Luther's Germany faced financial calamity.

1. General Outline

The article has two main parts. The first part (Sections 1-19), written with the goal of much earlier publication, is mainly theoretical. In it we find an introduction to tristimulus space, the embedding therein of the object-color solid (given a pre-defined illuminant spectrum), and the *optimal colors* on the exterior surface of the object-color solid (comprising reflectances that at all visible wavelengths have values of either 0 or 1 with at most two transitions between them). Some of the optimal colors, called *end colors*, are those with only one transition. (Visualizing the optimal-color surface as a clamshell, the end colors are where the two halves of the shell join.) Other optimal colors, called *full colors*, comprise the curve on the object-color solid that would just touch the minimum cylinder that contains that solid and is parallel to the achromatic axis. (If the object-color solid were a key and the cylinder were a lock, the full colors would be the points on the key that brush the inside of the lock.) All these constructs, and others in Luther's article, emerge most immediately from the theoretical work of Wilhelm Ostwald, and also owe some debt to works such as those of Erwin Schrödinger and Hermann Helmholtz. But Luther offers some new contributions as well. For example, he was probably the first to draw the shape of the object-color solid, and did so in several coordinate systems.

The second part (Sections 20-25) is a smorgasbord of material with a more practical flavor. In Section 20 we find a design of a template colorimeter and a discussion of filter colorimeters. Section 21 discusses a single-color sensor whose spectral sensitivity is compensated to the human luminosity sensitivity---a one-dimensional form of a tristimulus colorimeter or colorimetrically correct camera. Section 22 explains and endorses heterochromatic flicker photometry. Then Section 23 describes criteria for a trichromatic camera. Finally, Sections 24 and 25 wax philosophical about subjective versus objective colors: Luther saw subjective colors as illusions or perturbations on "real" physical colors---quite a contrast with more modern views.

In Part 1 Luther adds to Ostwald's optimal colors a set of mathematical relations among the optimal colors in tristimulus space. Perhaps the restriction of that mathematics to optimal-color reflectances has limited its familiarity in the color-science community.

Luther is credited, not for his optimal-color constructs, but for a rule for camera-sensitivity functions---that they should be nonsingular linear combinations of human color-matching functions to make the same color matches we do. A question for historians is how this statement is an advance over the earlier statements by James Clerk Maxwell [2] and Frederic Ives [3].

To give a flavor for and cast in modern terms some ideas in Luther's article, I choose one topic from Part I and one from Part II: I attempt to explain in modern terms the parallel-chord theorem of Section 11, and examine Section 23 in its own right and as it bears on the camera-sensitivity rule called by some the "Luther criterion."

2. An example from Part I: The parallel chord theorem

In Section 11 Luther poses a theorem that relates equal dominant wavelength and parallel chords in a particular color space. The English translation says, "The required wavelength range and the brightness of an optimal color can be read from the portion of the curved double scale located clockwise between the beginning and the end of the chord. A chord can be deconstructed into its two (vertical) primary moments, the ratio of which, according to absolute value and sign, indicates the direction and thereby the hue. Therefore, chords that, as vectors, are parallel represent optimal colors of the same hue." In context, it seems clear that by "hue" he means dominant wavelength. To me the proof needs much elaboration, so I offer it here.

The key to understanding the theorem is through Figure 9, a two-dimensional space in which appears a curve with several chords drawn---ostensibly parallel to each other. Thanks to conversations with Jan Koenderink of Utrecht, NL, the space, the curves, and the chords became clear:

a. The 2D space comprises two tristimulus dimensions---the exact choice doesn't matter, but black and white should be at the origin. As a geometric pictorial aid, you can imagine looking in parallel (not central) projection along the achromatic axis in 3D.

b. The curve in Fig. 9 represents one of the two loci (curves) of *end colors*. Together the long-end and short-end loci form a figure-8 shape crossing at the achromatic point: Half of the 8 comprises the long-end colors (up-transition wavelength parameter λ_1), and the other half comprises the short-end colors (down-transition wavelength parameter λ_2).

c. One might have expected a chord in Fig. 9 to connect one point from the λ_1 loop of the figure-8 and one point from the λ_2 loop. But Luther uses only one of the loops (e.g., the short-end-color loop), and finds both λ_1 and λ_2 on that loop. [One loop maps on the other (replete with wavelength labels) by a coordinate inversion.]

d. Given the above, here is the theorem: Let two pass-band optimal colors have transition-wavelength pairs (λ_1, λ_2) and (μ_1, μ_2), where $\lambda_1 < \lambda_2$ and $\mu_1 < \mu_2$. Denoting the unit step function by u, the (λ_1, λ_2) optimal reflectance as a function of wavelength λ is $u(\lambda - \lambda_1) - u(\lambda - \lambda_2)$, and the ($\mu_1$, μ_2) optimal reflectance is given by $u(\lambda - \mu_1) - u(\lambda - \mu_2)$. Integrating in λ with respect to the two illuminant-weighted color-matching functions, we get 2-vectors denoted by $\mathbf{x}$: $\mathbf{x}(\lambda_1) - \mathbf{x}(\lambda_2)$ and $\mathbf{x}(\mu_1)- \mathbf{x}(\mu_2)$. The various points $\mathbf{x}$ are the points on the short-end-color curve: $\mathbf{x}(\lambda_1)$, $\mathbf{x}(\lambda_2)$, $\mathbf{x}(\mu_1)$, $\mathbf{x}(\mu_2)$. The theorem is: If these two pass-band colors have the same dominant wavelength, the vectors $\mathbf{x}(\lambda_1) - \mathbf{x}(\lambda_2)$ and $\mathbf{x}(\mu_1)- \mathbf{x}(\mu_2)$---which are the directions of the chords between the λ's and between the μ's---are parallel to each other.

e. A proof of the theorem: The dominant wavelength of a tristimulus vector **X** is the wavelength λ' such that $\mathbf{X}(\lambda')$, **X**, and **W** are coplanar (where **W** is the tristimulus vector of white). That means $\det[\mathbf{X}(\lambda'), \mathbf{X}, \mathbf{W}] = 0$. If two tristimulus vectors $\mathbf{X}_L$ and $\mathbf{X}_M$ have the same dominant wavelength, then they are coplanar with each other and with **W**: $\det[\mathbf{X}_L, \mathbf{X}_M, \mathbf{W}] = 0$. Now, we have chosen tristimulus coordinates so **W** is nonzero only in its third component. Therefore the above determinant equation is equivalent to $0 = \det[\mathbf{x}_L, \mathbf{x}_M]$, where $\mathbf{x}_L$ and $\mathbf{x}_M$ are the 2-vectors in the dichromatic space. Substituting $\mathbf{x}(\lambda_1) - \mathbf{x}(\lambda_2)$ for $\mathbf{x}_L$ and $\mathbf{x}(\mu_1) - \mathbf{x}(\mu_2)$ for $\mathbf{x}_M$, the zero-determinant condition implies that the vectors are linearly dependent, and hence (being in 2D) must be parallel.

To me, the parallel chord theorem shows the originality of Luther's geometrical insight. However, because it deals only with optimal colors---which do not exist in nature---the theorem has not been widely used. It would be interesting to look in Luther's Part 1 for mathematical insights of more general applicability that have escaped modern attention.

3. An example from Part 2: Camera-sensitivity criterion

Section 23, on color photography, contains remarks on the spectral sensitivities of an ideal camera, and also on filter spectra required for a color-accurate additive synthesis of three images from three color-separated negatives. I will focus here on his criteria for camera-sensitivity functions because they have received a lot of citation and attention.

The roots of Luther's design of camera-sensitivity functions are to be found in the short paragraph that ends Section 20 and describes the design of a tristimulus colorimeter:

> "[...] it is possible to use undispersed light with suitable selective filters. In that case it is of course necessary that the spectral transmittance of T_λ of the filter, for example for the red stimulus determination, is in agreement with the following precondition: $T_\lambda \cdot E_\lambda = R_\lambda$."

Hence, the filter-detector combination ideally has the same sensitivity as a color-matching function R of the visual system.

In Section 23, the design of the tristimulus colorimeter is transferred with very little fanfare to the spectral sensitivities of a color-faithful camera. The message is that, to make the same matches as the eye, the camera must have spectral sensitivities close to those of the eye.

In the years that followed Luther's article, the principle he described evolved to a more precise form: In order for a trichromatic camera to match the same colors as the human observer, the spectral sensitivity functions of that camera must be linear combinations (and together must comprise a nonsingular linear transformation) of the human-observer color-matching functions. That criterion has variously been called the Maxwell-Ives criterion [4] and the Luther condition [5]. The term "Luther-Bedingung" was introduced by Manfred Richter [6], a student of Luther.

As brief as Luther's version of the camera-sensitivity criterion is, I will now make the case that it is an advance over the earlier versions by Maxwell and Ives.

Maxwell's 1855 version appears on pages 283-284 of [4]:

> "Three elementary effects, according to [Young's] view, correspond to the sensations of red, green, and violet, and would separately convey to the sensorium the sensation of a red, a green, and a violet picture; so that by the superposition of these pictures, the actual variegated world is represented." […]
>
> "This theory of colour may be illustrated by a supposed case taken from the art of photography. Let it be required to ascertain the colors of a landscape, by means of impressions taken on a preparation equally sensitive to rays of every colour."
>
> "Let a plate of red glass be placed before the camera, and an impression taken. […] Let it now be put in a magic lantern, along with the red glass, and a red picture will be thrown on the screen."
>
> "Let this operation be repeated with a green and a violet glass […] a complete copy of the landscape, as far as the visible color is concerned, will be thrown on the screen. The only apparent difference will be, that the copy will be more subdued, or less pure in tint, than the original. Here, however, we have performed the process twice---first on the screen, and then on the retina."
>
> "This illustration will show how the functions which Young attributes to the three systems of nerves may be imitated by the optical apparatus."

Maxwell clearly captured the imitation of color-matching functions by camera-sensitivity functions, and also recognized that the optical-projection part of the camera's function is separate and distinct, desaturating the rendered colors. However, he did not separate mathematically the light-sensing and projecting functions, nor did he capture the "linear transformation" idea of the mature camera-sensitivity criterion.

Ives's version of the criterion emerges by piecemeal examination of [3]:

> p. 13: "[The] new principle, first stated by me in a communication to this institute on November 21, 1888, is that of making sets of negatives by the action of light in proportion as they excite primary color sensations, and images or prints from such negatives with colors that represent primary color sensations."
>
> pp. 13-14: "[…] The eye is only capable of three primary color sensations […] According to Clerk Maxwell, the orange spectrum rays excite the red sensation more strongly than the brightest red rays, but also excite the green sensation [etc]; Maxwell's diagram is a graphic representation of the result of careful photometric measurements of the effect of the spectrum upon these primary sensations."
>
> p. 15: "The plates and screens [producing a photographic negative] are correct when they will secure negatives of the spectrum showing intensity curves substantially like the curves in Maxwell's diagram."

Ives clearly separated the sensing function of the camera from the rendering function, and attributed the imitation of visual sensitivity firmly to the former. The linear transformation idea was still not on the horizon.

Given these earlier works, it is not far-fetched to consider Luther's contribution a further step toward the mature criterion: a mathematical equation, and a precise though brief discussion of the equation. The linear-transformation freedom, however, is still not acknowledged.

Lest we consider that Luther missed an obvious point, it is instructive to see how later authors incorporated the linear-transformation freedom. That property was clearly understood by Neugebauer in 1956 [7]. But consider MacAdam's 1967 work [4]:

> p. 28: "Maxwell was quite specific about the required character of the controls. He said that three photographs should be made with spectral sensitivities proportional to the three spectral sensitivities had shown can be attributed to the eye to account for all of the infinite varieties of spectral distributions that can look alike (that is, 'have the same color')."
>
> In the next paragraph: "Thirty years later, Frederic Ives had much more suitable materials, and also the single-minded persistence required to carry out Maxwell's idea and to demonstrate its validity to a skeptical and even hostile audience. But his success has little influence on the development of modern color photography. Through his son Herbert, however, it had a significant influence on the modern technology of color measurement that very effectively supplements spectroscopy (spectrophotometry) in all industries that supply colored products."

Although MacAdam's words are accurate, the statement "proportional to the three spectral sensitivities" leaves unexpressed the freedom of linear transformation. It is a subtlety that gained significant attention only when computers became commonly available to exercise the linear transformation freedom.

I found it quite educational to examine in this way the contribution of Maxwell, Ives, and Luther of the camera criterion that bears various combinations of their names. The earliest inventors of an idea did not necessarily grasp or express its complete essence. There may even be a fragment today that we are not using to complete advantage.

4. Conclusions

This introduction has been a whirlwind tour of Luther's article followed by two highly focused micro-views. It cannot be said that I have "stolen the thunder" of the article or given away the ending. Luther's article is the work of a scholar of great accomplishment but terse expression, who has summarized color science as of the third decade of the 20th Century in one diverse essay. Enjoy!

Michael H. Brill
Datacolor

[1] R. Luther, Aus dem Gebiet der Farbreizmetrik, *Z. Technische Physik.*, **8**, 540-558 (1927). [English translation by Rolf G. Kuehni, 2009].

[2] J. C. Maxwell, Experiments on Colour as perceived by the eye with remarks on colour blindness. Edinburgh Royal Society Transactions. XXI, 1857, pp. 275-298. [1855]. Scientific Papers Vol. 1 pp. 126-154. See pp. 283-284 for the statement about camera sensitivities.

[3] Frederic E. Ives, Photography in the colors of nature, *J. Franklin Inst.* **131**, 1-21 (1891).

[4] David L MacAdam, "Color Science and Color Photography", Physics Today, January 1967, pp. 27-39.

[5] Francisco Martínez-Verdú et al., "Concerning the calculation of the color gamut in a digital camera," *Color Research and Application* **31**, 399-410 (2006).

[6] M. Richter, I. Schmidt, A. Dresler, *Grundriss der Farbenlehre der Gegenwart*, Dresden: Theodor Steinkopff, 1940, p141.

[7] H. E. J. Neugebauer, "Quality factor for filters whose spectral transmittances are different from color mixture curves, and its application to color photography," *J. Opt. Soc. Amer.*, **46** (1956), 821-824.

On color stimulus metrics

by R. Luther, Dresden

Zeitschrift für Technische Physik, 1927, Vol. 8, Nr. 12, 540-558

Contents: In the first part, by expanding classical color theory and use of the unit 'color moment', new parameter triads are developed and used to produce three-dimensional color solids, particularly useful to represent the chromatic properties of object colors. In the second part, related to color measurement technology, heterochromatic photometry, and three-color photography are discussed.

1. Given time limitations, I am unable to provide a logically developed, comprehensive survey but – as already the title of my presentation indicates – only more or less loosely arranged excerpts from the field of color science. Any such selection is indicative of personal preferences and for this reason I would like to briefly describe the foundations of the present one.

I would like to particularly emphasize that my expositions and reflections have their source in practical, technical questions of my field of specialty, which lead into the same questions: theory of color photography; of anaglyphs, of orthochromatics, and its opposite pole: darkroom illumination, methods of color measurement, and of heterochromatic photometry. In agreement with the title of my presentation I will concern myself almost exclusively with what v. Kries names color stimuli – in contrast to the color perceptions that they cause. I will in general remain in the purview of Schrödinger's basic colorimetry,[1] but also make use of the term heterochromatic brightness. I will not concern myself with the so-called theories of color vision – Helmholtz, Hering, v. Kries. I will attempt to present preferably unknown or, to my best knowledge, little know subjects. I will reach back into classical representations of color theory only to the extent that this is absolutely necessary, all the more because recently several summaries[2] of that theory have been published.

If, according to Ostwald, we term color stimuli that under identical conditions result in identical percepts but objectively have different spectral compositions as "metamers," we can define classical color theory in an abbreviated way as the theory of the generation of metameric – chromatic as well as achromatic – color stimuli.

2. It is known that under certain limiting conditions: cone vision, average color-normal eye, limitation to "reduced" colors in the sense of D. Katz[3] – the totality of here applicable relationships is of threefold diversity. A given color stimulus can therefore be completely defined by its properties in terms of three suitably defined independent parameters.

Once a triad of parameters has been selected it is, of course, possible to derive from them an arbitrarily large number of new triads that express the same complex of facts with identical completeness and that are equally suitable for the definition of a color stimulus.

The first part of my presentation concerns itself, among other things, with some as yet not employed parameter triads.[4] The selection, out of a limitless number of possibilities, has been made less for the purpose of achieving maximal generality but rather is based on a point of view of practicality. Only systems that are sufficiently informative, systems that make it possible to comprehend as many relationships as possible in the simplest possible manner, those that simplify practical work, and finally – as much as possible – demonstrate new connections are to be considered. I will not mention closely related derivations and generalizations.

3. My basis is the well-known Young-Helmholtz-Maxwell, so-called trichromatic, system. In that system every color stimulus, arbitrarily composed from spectral stimuli, is unambiguously defined by the "amounts" of three independent, but otherwise arbitrarily selectable, elementary or – according to v. Kries – "primary" stimuli [*Eichreize*].

As all mixed stimuli, in the final analysis, are composed of spectral stimuli the first step is a "spectrum standardization" (v. Kries). Figure 1 top shows such a "standardization" of the daylight spectrum,[5] as first determined by Maxwell. The abscissas represent the wavelengths of the standard spectrum, the ordinates the amounts of the *R, G,* and *B* of the three selected primary stimuli Red, Green, and Blue, as well as their sum $S = R + G + B$, for identically narrow spectral ranges of always 10 nanometers.

The amounts of the three primary stimuli, together resulting in white, are as usual of identical magnitude, each equaling 1000 stimulus units. As a result, white composed from the total spectrum, independent of its absolute brightness, has the stimulus sum of

$$\mathcal{S} = \int_{700}^{380} S(\lambda) \cdot d\lambda = 3000 \text{ stimulus units}$$

The identity of the amounts of the three primary stimuli in white is expressed in the areal identity of the areas between the three primary functions and the abscissa axis in the top part of Fig. 1:

$$\mathcal{R} = \int_{700}^{380} R(\lambda) \cdot d\lambda = \mathcal{G} = \int_{700}^{380} G(\lambda) \cdot d\lambda = \mathcal{B} = \int_{700}^{380} B(\lambda) \cdot d\lambda = 1000.$$

In case of an arbitrary mixed stimulus, i. e. a mixture of spectral stimuli such as that of the emerald [*Smaragd*] measured by Exner, as illuminated by daylight reflected into the eye, the amounts of the three primary stimuli can, as is well known, be easily calculated from the reflectance spectrum (Fig. 1 bottom). To achieve this, for a sufficiently large number of wavelengths the ordinate values of the primary stimuli have to be multiplied with the fractions of the illuminating light $Rm(\lambda)$ reflected at the same wavelengths and the size of the resulting area determined, e.g. for the red stimulus.

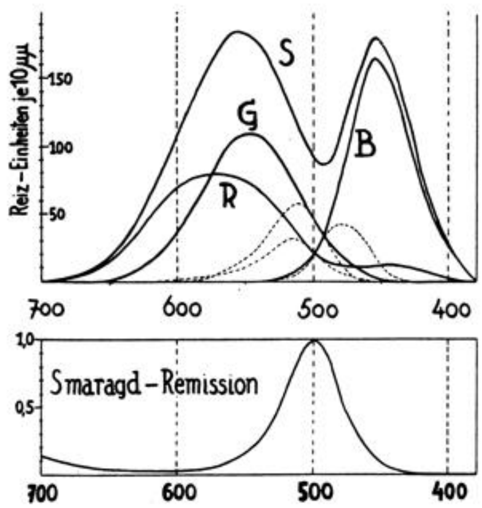
$$\mathcal{R}_{\mathrm{Sm}} = \int_{700}^{380} R(\lambda) \cdot Rm_{Sm}(\lambda) \cdot d\lambda .$$

4. When many sets of standard stimuli have to be calculated it is possible to simplify the work, e.g. when determining R_{Sm}, by arranging the spectrum on the abscissa axis in such a way that equal increments of the abscissa correspond to equal increments of the red stimulus in the daylight spectrum. To achieve this, one integrates the *R* function of Fig. 1

relative to λ and in this way obtains, as shown in Fig. 2, the required division of the spectrum. The same procedure is applied to the *G* and *B* functions and, for control purposes, possibly also to the *S* function.

If in the once and for all determined coordinate systems the unit of the ordinate is selected to be the same as that of the reflection spectrum, the ordinates of such a spectrum can be transformed without recalculation. The resulting areas[7] correspond now to the amounts of the three primary stimuli. Figure 3 shows this procedure for the reflectance spectrum of the emerald in schematic form.

For the present case the result is that the mixture stimulus caused by the emerald consists in total of 713 stimulus units, divided into 198 *R* units, 317 *G* units, and 198 *B* units.



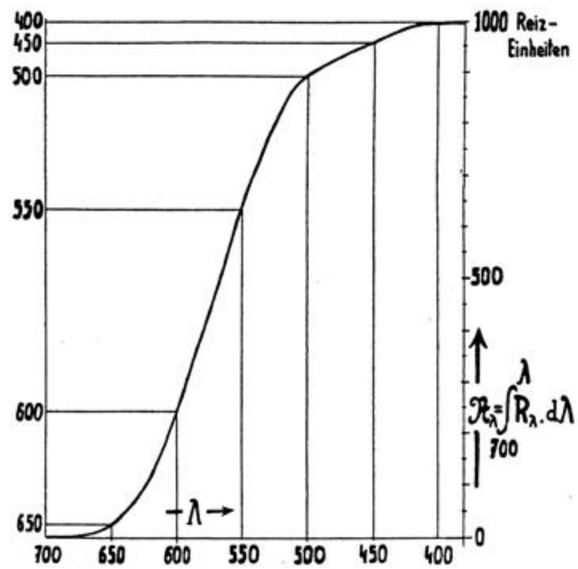


Fig. 1. (left) Calibration of the daylight spectrum and determination of the three primary stimulus amounts in the spectrum of a mixed stimulus (reflectance of an emerald).
[Reizeinheiten je 10 μμ: stimulus units per 10 nm; Smaragd_Remission: reflectance of an emerald]

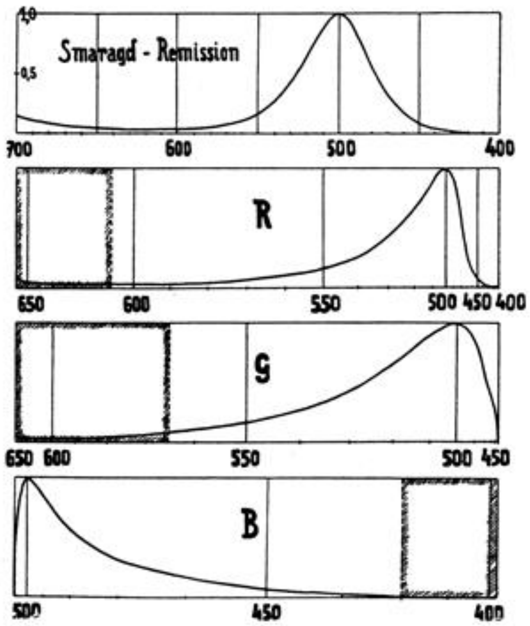
Fig. 2. (right) Separation of the spectrum according to equal increments of the red stimulus.

5. These numbers provide a complete but not easily comprehended picture of the characteristic appearance of the color of the emerald. Comprehension is improved by plotting the components of the three primary stimuli of spectral as well as mixed stimuli such as that of the emerald:

$$r = \frac{R}{S}, \quad g = \frac{G}{S} \quad \text{and} \quad b = \frac{B}{S}$$

as triangular coordinates into an equilateral triangle (Fig. 4 top). For the emerald the values are $r_{Sm} = 0.28$; $g_{Sm} = 0.44$; $b_{Sm} = 0.28$.

Plotting the spectral stimuli into the chromatic diagram has, in terms of Newton's center of gravity construct, the following easy-to-grasp significance: Assume that red, green, and blue weights proportional to the amounts of the corresponding ordinate values of the three spectral primary stimuli of Fig. 1 hang at every 10 nm interval along the spectral curve in the chromatic diagram. The result is that the system supported at the white or achromatic point *U* [*Unbundt*] – the center of gravity – is in balance. The weight pulling on the balance point is equal to the sum of all individual weights (i. e. the stimulus sum of white). The weights at each spectral wavelength can be removed and re-attached at the corresponding corner points without upsetting the balance. The white or achromatic point corresponds to a state of balance of the chromatic spectral stimuli, where they compensate each other, with none of them predominating.



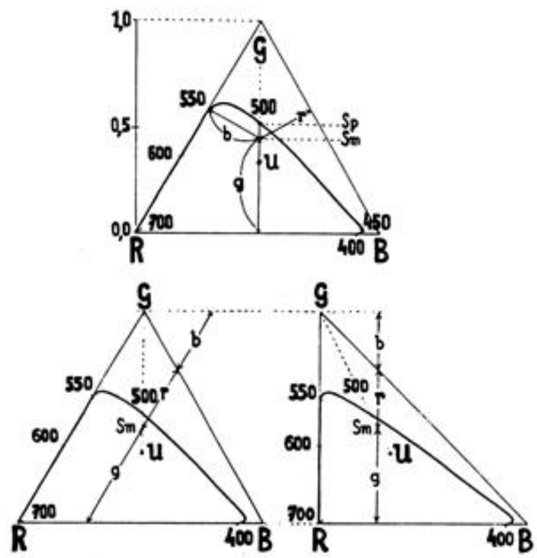


Fig. 3. (left) Determination of the amounts of the three primary stimuli in the reflectance spectrum of the emerald.

Fig. 4. (right) Locus of the emerald stimulus in the “color diagram” based on the three primary stimulus components. Parallel projection of the color diagram.

Every chromatic mixed stimulus has a corresponding center of gravity or balance point in the color diagram, located between the achromatic point and the spectral curve (including the straight connecting line from λ = 380 nm to λ = 700 nm corresponding to purple colors).

Another kind of data representation is shown at the bottom of Fig. 4. It is particularly convenient if the color diagram – which is valid without distortion of the Newtonian center-of-gravity equilibrium, and of which we will make further use below – is transformed using parallel projection – in the present case into a right triangle (Fig. 4 bottom right).

The location of the mixed stimulus in the color diagram indicates that its hue corresponds to the wavelength of λ = 500 nm[8] and, in the sense of the Newtonian center of gravity construct, represents the mixed stimuli of the green spectral stimulus (*Sp*) of λ

= 500 nm and of the achromatic center (*U*). The relative amounts of spectral and achromatic stimuli in the total stimulus is obtained from the lever ratios

$$\frac{U - Sm}{U - Sp} \quad \text{and} \quad \frac{Sm - Sp}{U - Sp} \qquad \text{(see Fig. 4 top).}$$

For the emerald the result is that of the sum total of 713 stimulus units of the complete stimulus 40% are due to the achromatic stimulus and the remaining 60% due to the spectral stimulus λ = 500 nm. The latter component 0.60 is called "saturation," or "color saturation" (in stimulus units).

By reporting hue, saturation, and "intensity", i.e., the total weight in the Newtonian sense, the totality of the stimulus in stimulus units, we obtain the Grassmann-Helmholtz parameter triad [9], thereby resulting in an unambiguous and easy to comprehend definition of the color stimulus.[10]

6. Unsatisfactory and distracting is the calculation of "intensity" and of saturation in stimulus units, leftovers from a particular trichromatic system not otherwise present in the Grassmann-Helmholtz system. Already early (Grassmann), the need arose to replace the arbitrarily selected stimulus units as a measure of intensity with an experimental, independently determinable unit of measurement: brightness (more correctly: surface brightness). Here I do not want to concern myself with v. Kries' critique of the term heterochromatic brightness, as well as Schrödinger's related comments (note 1), but rather limit myself to the experimental data.

In regard to the dependence of specific brightness H_λ of the spectral stimuli on wavelength, all methodologies of heterochromatic brightness determination used by Ives' in his systematic experiments,[11] with one exception,[12] essentially agreed in the final result, if with very different variability. An additional result was that on a practical level, i.e., within the unavoidable errors of measurement and individual variation, the brightness values measured in this manner are purely additive.[13]

If one assumes the existence of a brightness $\mathcal{H}$, measurable independent of hue and saturation, and with the additive property $\mathcal{H} = \int H_\lambda \cdot d\lambda$ in stimulus summation, it follows that brightness $\mathcal{H}$ of an arbitrarily composed stimulus can be calculated from the amounts of the reference stimuli contained in it, using a linear formula:

$$\mathcal{H} = \rho\mathcal{R} + \gamma\,\mathcal{G} + \beta\,\mathcal{B}.$$

It should be noted here that in case of the usual choice of the three reference stimuli (red, green, and blue) the factor β – the specific brightness of the blue reference stimulus – is always very small compared to ρ and γ.

7. What are the resulting changes for the spectral reference functions and the color diagram?

In case of the spectral reference functions the ordinates obviously have to be multiplied with the corresponding coefficients ρ, γ, and β. As a result, the stimulus sum function *S* takes on the form of the brightness sum function, i.e., the so-called spectral brightness function *H* (Fig. 5 bottom).

The change in the color diagram can be represented using the following idea that will later come in handy in another form.

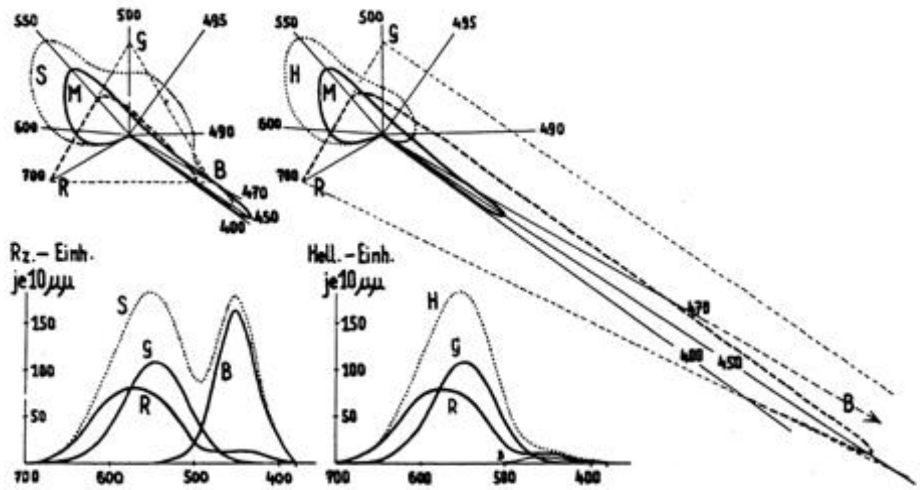


Fig. 5. Schematic representation of the change in the color diagram (top) and the primary stimulus curves (bottom) when changing from calculation in stimulus units to one in brightness units. The polar moment curve *M* remains unchanged. The stimulus sum curve *S* changes to the brightness curve *H*. The dashed parts of the two upper figures are related according to the ratio of central perspectives.
[Rz.-Einh.: stimulus units; Hell.-Einh.: brightness units]

The ("leucocentric"[white-centered]) lines drawn in intervals of 10 nm from the achromatic point to the spectral curve of the color diagram can, in the sense of a Newtonian center of gravity construct, be viewed as a system of solidly connected levers, supported at the achromatic point, at the ends of which hang as weights the stimulus sums of the individual 10-nm spectral intervals.

If the Newtonian center of gravity is not to be disturbed when replacing stimulus units with brightness units, neither for the spectral stimuli nor for the reference stimuli, this can, seemingly in the simplest way, be achieved as follows: all levers (= leucocentric lines) without change in direction and corresponding to the new weights attached to them are also changed in their length in a related manner so that the product of length of lever and weight, i.e. the leucocentric lever moment, remains unchanged.

Figure 5 schematically shows this change from counting the weights in stimulus units to counting them in brightness units. The change in the three spectral primary stimulus curves and the change from the stimulus sum curve *S* to the spectral brightness curve *H* is indicated on the bottom – the corresponding change in the color diagram is shown on top.[14] It is obvious that the levers to which the blue stimulus is attached, while not changing in direction, are increasing in length, corresponding with the very small "specific weight" of the blue stimulus.

The spectral lever moments remain unchanged in this transition from one color diagram to the other and are drawn leucocentrically polar with full line *M* (of retort shape) in both color diagrams.[15] However, the stimulus sum curve *S*, shown also in polar form with a finely dotted line in the upper left diagram, changes to the correspondingly represented spectral brightness curve *H*[16] in the right color diagram.[17]

The transformation coefficients *ρ, γ,* and *β* depend on the choice of the primary stimuli. Depending on the conditions, the always very small coefficient *β* can have a value of zero or even be negative.[18] My three primary stimuli have been chosen in a manner that $\beta = 0$,[19] i.e. the infinitely small weight of the blue primary stimulus is attached to an infinitely long lever: the lever moment does not change its finite value.

Relative to the new brightness values the characteristic numbers for the emerald in the Grassmann-Helmholtz "monochromatic" system are: brightness = 0.26 of the total spectrum of 515 brightness units, saturation = 0.63.

8. Introduction of the moments allows construction of a new triad of parameters that offers, in addition to ease of comprehension, many advantages.

It is possible to relate the Newtonian center of gravity of a mixed stimulus to two axes rather than to three points, the loci of the three primary stimuli in the color diagram. The two axes are not parallel in the plane of the color diagram and the torques applied by the individual weights on them can be made the basis of the calculation.[20]

In agreement with the psychological arrangement of the hues – the so-called four-color system – I place the two axes through the achromatic point in the direction Green-Purple and Blue-Yellow and, to aid comprehension and in agreement with Fig. 6, change the obliquely angled axes into right-angled ones using parallel projection (whereby the weights remain unchanged). If we now consider each leucocentric moment to be reduced to its right-angled components, that is into the torques for the two axes Green-Purple and Blue-Yellow we obtain the following relationships.

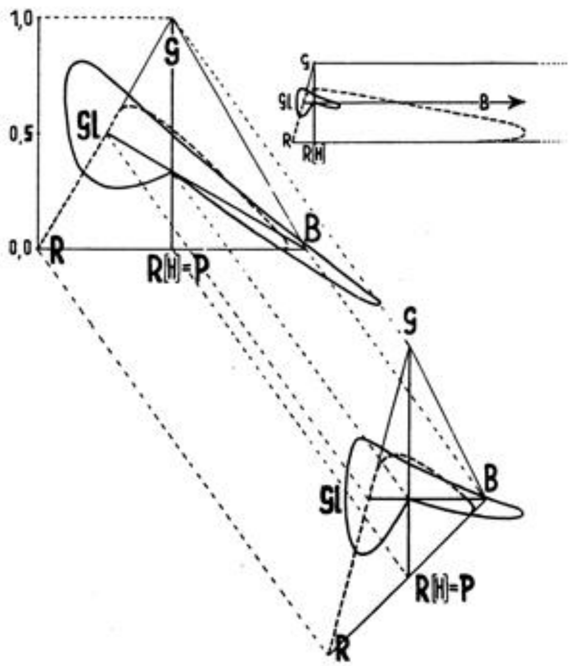


Fig. 6. Parallel projective transformation of the equilateral color diagram with tilted moment axes into a color diagram with rectangular moment axes. Additional transformation into a color diagram based on brightness units (top right, reduced scale). The specific brightness of the blue stimulus is equal to 0.

The torques of all weights that, for example, are applicable on the right side of the Green-Purple axis correspond in their totality to a – in two words – bluish torque, or the "blueness" – to use one of Hering's expressions. Opposed to this blueness are the "yellowish" torques of the weights active on the left of the Green-Purple axis, the total torque of which – again after Hering – can be termed the "yellowness" of a mixed stimulus. If yellowness exceeds blueness the total moment corresponds to a yellowish hue, and vice versa. If both are identical, that is the total moment is zero, the result corresponds to a hue that is neither bluish nor yellowish and can only lie on the Green-Purple line. What is decisive is the difference: blueness minus yellowness.

The identical idea can be applied to the to Green-Purple moments acting on the Blue-Yellow axis and one can speak, in agreement with Hering, of greenness-redness, that is, greenness minus redness. (Hering's Red, less yellowish and more purplish than the primary red that I also call Red, will be distinguished from the latter by the addition of [H].

The two moments Greenness-R[H]edness and Blueness-Yellowness, together with the "weight" of a stimulus, either expressed in stimulus or in brightness units, form a new parameter triad[21] with which a given color stimulus can be defined uniquely and in a manner that is quite easy to comprehend.

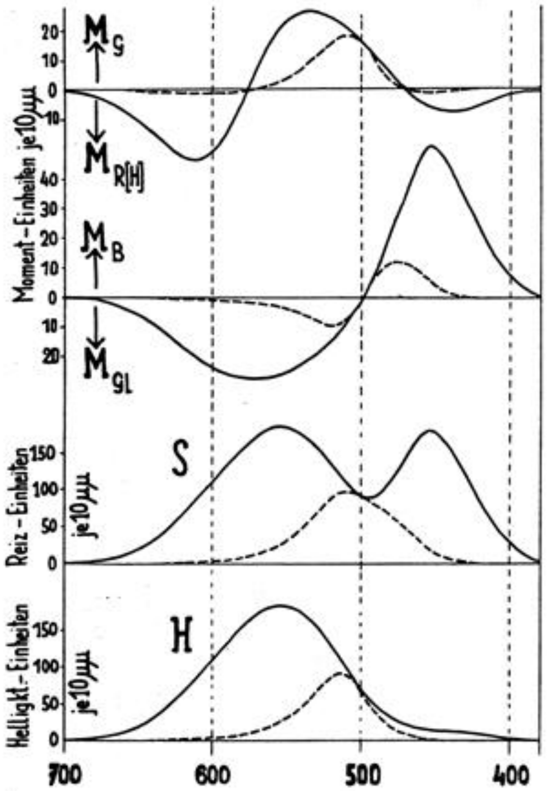
Of course, the two torques add up like vectors and, with the sign and the amount of their ratios, show their direction in the color diagram, thereby showing the hue of the resulting leucocentric color moment in a comprehensible manner. In an analogous manner one might call the result "colorfulness." By dividing the leucocentric color moment by the "weight" of the stimulus the length of the leucocentric lever is obtained, from which its saturation and the shares and amounts of the three primary stimuli are immediately available. This method represents the reciprocal conversion key of the tetra-tri-, and monochromatic systems.

9. In case of a composite mixed stimulus, for example an absorption or reflectance spectrum – such as our emerald spectrum – we can as before (paragraph 3) extract the two reference moments as well as the total stimulus or the brightness, respectively. To achieve this, first the amounts of the two reference moments are plotted at 10 nm intervals as ordinates in the standard spectrum. The resulting curves (Fig. 7) can be calculated from the corresponding amounts of the reference stimuli according to the linear formula[22] $M = mR + nG + pB$. (The area between the abscissa axis and the R(H)ed moment curve, that is, the sum of all spectral R(H)ed moments is of the same size but opposite sign compared to the corresponding area of the Green moment curve and the same applies for the areas under the Blue and Yellow moment curves because the two reference moments of the total achromatic spectrum equal 0.)

When multiplying, for a sufficiently large number of wavelengths, the ordinates of this curve with the reflectance values of the emerald and calculating – considering the sign – the total area, the two total moments of the emerald stimulus are obtained. The same applies to the stimulus sum curve *S* and the brightness curve *H*, respectively.[23]

On making these calculations (see dashed lines in Fig. 7) according to the parameter values I selected, the following characteristics of the color of the emerald are obtained: Blueness-Yellowness = 0; Greenness-R(H)edness = 79 moment units; weight =

713 stimulus units (= 0.24 of the total spectrum); brightness = 513 units (= 0.26 of the total spectrum).



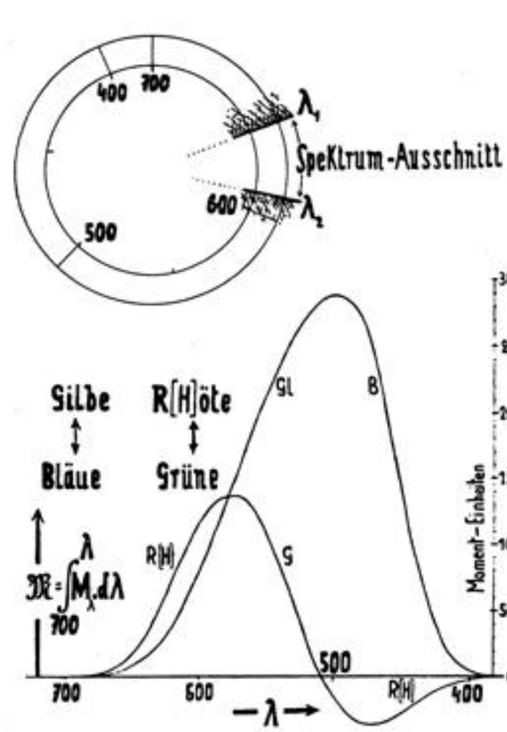


Fig. 7. (left) Spectrum calibration according to the two primary moments, stimulus sum, and brightness. Determination of the related magnitudes in the reflectance spectrum of the emerald. [Moment-Einheiten je 10 μμ: moment units per 10 nm]

Fig. 8. (right) "Optimal object colors" as mixed stimuli of continuous unreduced spectrum sections. Spectrum calibration according to the two primary moment integrals $\mathcal{M}$ for the determination of the two primary moment sums of the optimal object colors. [Spektrum-Ausschnitt: spectral section, wavelength range; Gilbe: yellowness, Bläue: blueness, R(H)öte: redness according to Hering, Grüne: greenness]

10. A special case of summation of spectral stimuli is of particular interest and results in a simple and easy to comprehend geometric representation. This applies to spectral stimuli representing a continuous, unreduced portion of the total normal spectrum.

To be able to view all possible cases we imagine the spectrum as a closed circle (Fig. 8). Two radii, imaginable as slit constraints, separate depending on their position a continuous, unreduced portion of the spectrum, of choice width and choice location, between wavelengths λ_1 and λ_2. The fact that all spectral sectors of this kind have properties of maximality, empirically found by Ostwald, in the ideal case obtainable from transmittance or reflectance spectra of pigments with very steep and limited absorption ranges, was substantiated by Schrödinger[24] as follows: They correspond to stimuli that at a given level of saturation have the highest brightness or at a given level of brightness the highest saturation. (In what follows they will be named optimal object color stimuli, abbreviated optimal colors.) All other object color stimuli can be obtained by darkening the optimal colors.

The two wavelengths λ_1 and λ_2 together with the absolute, related to the total spectrum, or the relative, related to optimal color, "intensity" – for example in brightness units – offer again a triad of parameters sufficient for the complete description of a color stimulus[25] and for the reproduction of that stimulus with a suitable apparatus – for example the Maxwell-Ostwald color mixer. Here I would like to provide the data for the emerald: λ_1 = 561 nm, λ_2 = 458 nm, brightness = 0.50 of the optimal color = 0.26 of the total spectrum.

The geometric representation of the triad: brightness, λ_1 and λ_2 – perhaps to be called the "dichromatic system – is not easy to comprehend, however the following considerations point the way to a representation that is easy to comprehend.

11. Because moments of the same directionality add up like scalars, one can calculate the two moments $\mathcal{M}$ of a section of the spectrum by summing or integrating, respectively, the two spectral moments M_λ from the beginning of the spectrum, that is from λ = 700 nm, first until λ_2 and then until λ_1, followed by calculation of the difference

$$\mathcal{M}(\lambda_1..\lambda_2) = \int_{700}^{\lambda 2} M_\lambda \cdot d\lambda - \int_{700}^{\lambda 1} M_\lambda \cdot d\lambda .$$

Figure 8 bottom shows these moment sums or moment integrals from λ = 700 nm to λ as functions of λ in the form of curves. The difference between two ordinate values results for each of the two curves in the moment of the spectrum section enclosed by the two abscissa sections beginning at the end of the spectrum.

If we now plot the ordinate values belonging to identical wavelengths of the two curves into a coordinate system, that in our case is rectangular, we obtain a curve with gradation in wavelengths as well as in brightness (or stimulus sum) units (Fig. 9). We can name it the (spectral) moment sum curve. This curve has easy to comprehend, practically important properties.

All color stimuli are represented by vectors, the optimal colors by curve chords. Portions of the chords of optimal colors correspond to non-optimal stimuli. They can be seen as a weakened optimal colors, for example that of the emerald *Sm*. The required wavelength range and the brightness of an optimal color can be read from the portion of the curved double scale located clockwise between the beginning and the end of the chord.

A chord can be deconstructed into its two (vertical) primary moments, the ratio of which, according to absolute value and sign, indicates the direction and thereby the hue. Therefore, chords that, as vectors, are parallel represent optimal colors of the same hue. The hue can be read at the locus of the parallel tangent (infinitely narrow wavelength range), or in the corresponding color diagram (Fig. 9 right).

The chord length – the resultant of the two reference moments – is the "color moment" or the "colorfulness" of the optimal color. Complementary optimal colors have opposite numerically identical color moments. For every optimal color $a \rightarrow b$ there are therefore two complementary optimal colors $b \rightarrow a$, and $c \rightarrow d$. Complementary spectral colors, that is, optimal colors of infinitely narrow wavelength range, therefore have parallel tangents of opposite direction.

12. Parallel shifting of a chord can, according to the above, be useful for the convenient determination of the spectral width in case of "isochromatic" widening of the width, that is, while maintaining the same hue. In case of a parallel shift of this kind the chord length, that is, the colorfulness or the color moment, changes from the value zero (infinitely narrow spectral width) via a maximum *V* all the way again to zero (total spectrum). The maximal chord length, that is the maximal "colorfulness," corresponds to Ostwald's full colors, because when maximal values are attained the curve is sectioned either at two complementary points with parallel tangents or in the corner (see figure).

The chord length, the color moment, or the colorfulness can be made comprehensible by using the following consideration. If we take the lightness of an optimal color to be, in the Newtonian sense, its "weight," the following relations apply:

Moment of optimal color/weight of optimal color = length of lever of optimal color in the color diagram.

Length of lever of optimal color/length of lever of spectral color = saturation, i.e., share of spectral color in the optimal color.

Weight of optimal color · saturation = weight of spectral color in the optimal color.

Therefore:

Moment of optimal color = length of lever of the spectral color · weight of the spectral color in the optimal color.

The chord length (= moment of the optimal color) is therefore proportional to the amount of spectral stimulus in the optimal color stimulus. The proportionality factor, the lever of the spectral stimulus, naturally depends on hue and the selected amounts and measurement units. But it is constant for a give hue, that is, for a group of parallel chords of equal direction.

Ostwald's full colors are, in the sense of classical color metrics, distinguished by containing the maximum amount of spectral stimulus that the most perfect pigments can have: as a result, also in the sense of classical color science they represent "exceptional cases."

13. The moment sum curve of Fig. 9 (see also Figs. 10 and 13) contains a few additional chords that show an example of the application of the spectral moment sum curve when solving a practical problem. A colorant with the two complementary spectral ranges *ab* and *cd* is a so-called invalid gray. The same is true for another colorant with the spectral ranges *bc* and *da*, because it also results in an invalid gray, one that is "complementary" to the first one. Anaglyphs produced using such "complementary' false-gray pigments would offer an interesting kind of image: when viewed with a gray set of glasses, out of a confusion of gray lines an object would begin to appear in three dimensions. A. Callier[26] has produced and demonstrated such complementary false-gray filters. To a high degree

they have the annoying property of making the yellow spot entoptically visible. In addition, their color depends very much on the spectral composition of the light as well as on the adaptation and the individual properties of the viewing eye. All these properties can be used for measuring purposes.

Much more might be said in regard to this “moment sum curve.” However, while making use of it, I will limit myself to the construction of three solids particularly suitable to represent the properties of object colors.

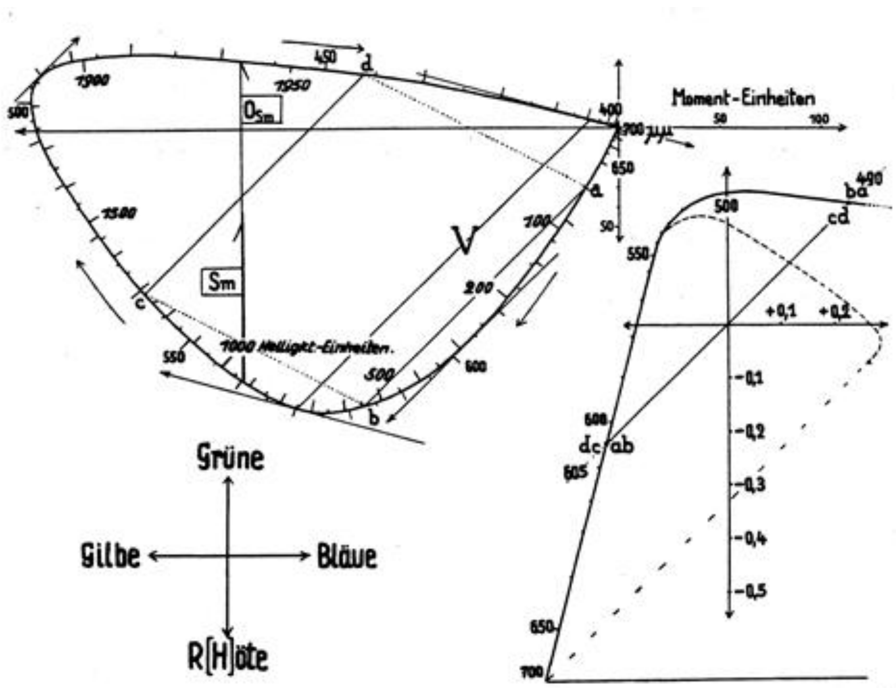


Fig. 9. Moment sum curves with dual scales in wavelength and brightness units and their relationship to the color diagram. Vectorially parallel chords correspond to optimal object colors of the same hue. The portion of the double scale located clockwise between beginning and end of a chord indicates the open region of the wavelength range. The cord length is equal to the “color moment” and proportional to the content of spectral stimulus. Ostwald’s full colors *V* are imbued with maximal color moment and spectral stimulus and section the curve either at complementary wavelengths, i. e., at points with parallel but opposite directed tangents, or in the corner. Representation of two “complementary” invalid gray filters for stereoscopy. Representation of the emerald stimulus as weakened optimal object color.

14. Figure 10 shows sections through a solid that correspond to four different hues. The corresponding three rectangular coordinates of the solid represent hue, saturation, and brightness. Because in each of the shown sections hue is constant they are variable in regard to saturation and brightness. Saturation is represented by the ordinate, brightness by the abscissa, counted from the left incline of the (solid) curve toward the right. On the left the inclining curves represent brightness zero. Optimal colors are located on the right declining curve. Their brightness is represented by the horizontal distance from the left “black” incline, their saturation by the ordinate. In addition, the two limiting wavelengths of the optimal colors can be read from the intersection points of the horizontal lines (of equal saturation), projected with vertical lines onto the abscissa axis, and the

corresponding wavelengths (see curve for Green-Blue 490). For this purpose, the abscissa axis has an equiluminant wavelength scale (see Note 23), the start of which obviously depends on the hue represented by the section.

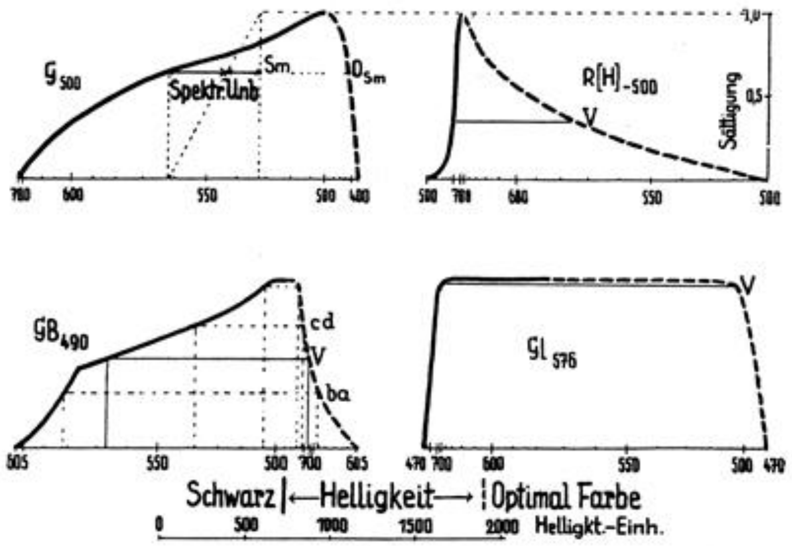


Fig. 10. Four sections through a three-dimensional solid valid for object colors, with the coordinates hue, saturation, brightness. The abscissa is scaled by an equiluminous spectrum; the left inclines correspond to brightness equals 0, the dashed curve section on the right contain the optimal object colors. Representation of the emerald color *Sm* as a darkened optimal object color O_{Sm}. Graphic determination of the amount of spectral stimulus in the emerald stimulus. The maximal spectral stimulus is contained in optimal object colors. (Section Green-Blue 490). Exceptional situation in case of yellow optimal object colors.
[Spektr.: spectral stimulus, Unb.: achromatic stimulus, Sättigung: saturation, Schwarz: black, Helligkeit: brightness, Optimal Farbe: optimal object color stimulus]

The brightness of the spectral portion of an optimal color equals the product of its brightness and saturation. This brightness of the spectral component is represented by the areas of the rectangles shown in the curve Green-Blue 490. For narrow wavelength ranges the area is small but grows with decreasing saturation and increasing brightness due to the widening of the section. A maximum is reached in case of Ostwald’s full colors. Further widening of the wavelength range results in reduction of the area all the way to the situation where the section encompasses the complete spectrum (saturation = 0) and the area is equal to zero.

Non-optimal color stimuli (for example that of the emerald *Sm*), they can be considered to be darkened optimal color stimuli, are located on the horizontal line corresponding to their optimal color (O_{Sm}) in the direction of the left (black) incline (see curve Green 500). Horizontal lines, therefore, are lines of constant saturation and constant hue, that is, of constant color character, respectively chroma, respectively stimulus type; they are identical with Ostwald’s “shadow series”. In case of the emerald stimulus, geometrically as a matter of course, its total brightness has been divided into the brightness of the spectral (Spektr.) and the achromatic (Unb.) portions.

From the different shapes of the four curves one can deduce the changes in relations in different hues in regard to changes in brightness and saturation. It is evident that yellow object colors represent a special case. Here the full color, with a brightness of over 85% of the total spectrum is nearly (≈98%) fully saturated, contrary to R(H)ed, where the full color is very dark and has little saturation. It is further evident that if one of the limiting wavelengths passes through the end of the spectrum the result is a bend in the ascending, opposite end of the curve (curve GB 490).

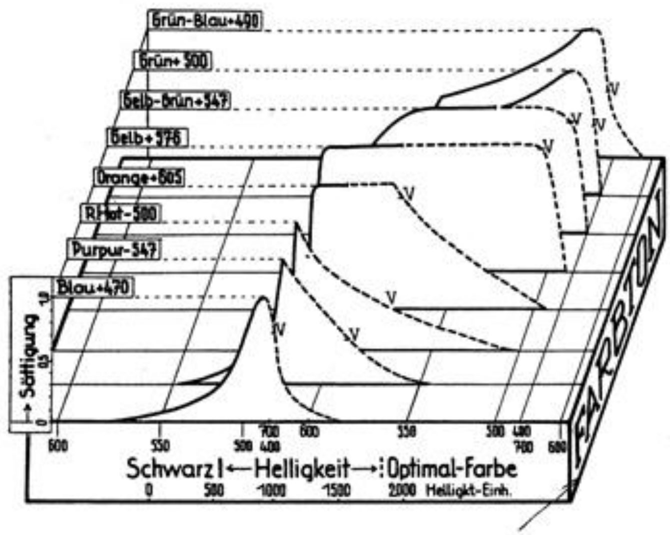


Fig. 11. Parallel-perspective representation of the solid discussed in Fig. 10.
[Farbton: hue]

Figure 11 is a perspectival representation of a series of sections located above two connecting equiluminous spectral scales.

15. Another "object color stimulus solid" with the rectangular coordinates Blueness-Yellowness, Greenness-R(H)edness, and stimulus sum in one case, brightness in the other, is shown in Figs. 12, 13, and 14. On top of Fig. 12 and in Fig. 14 the vertical scale is in stimulus units; in Fig. 12 bottom and Fig. 13 the vertical scale is in brightness units. The shape of the solid is reminiscent of a parallelepiped with rounded corners and usually positively rounded planes. Its center of symmetry is binomial.

The null point of the coordinate system resides in the achromatic "black point," having zero stimulus or brightness. The achromatic axis, having a color moment of zero rises from black via gray to white. A color stimulus corresponds to every point not on the axis but within the solid. Its stimulus sum, respectively brightness, can be read from its vertical coordinate, its hue from the ratio, and its color moment from the resultant of the two horizontal coordinates. Correspondingly, the points of highest colorfulness at a given level of brightness, respectively stimulus sum, are located on the surface of the solid. The points representing all other object color stimuli are located in its interior.

In the additive mixture of two stimuli their Blue-Yellow moments, their Green R(H)ed moments, and their brightnesses, respectively stimulus sums, are added up. In other words: The vectors drawn from the point of origin to the two points are added up in a vector parallelogram. Therefore, every stimulus can, in the sense of the Grassmann-

Maxwell vectorial representation, be seen as a vector (see also Pilgrim op. cit.). In case of proportional mixture of two stimuli (for example in case of disk mixture) the resulting mixed stimuli are located on the straight line connecting the two stimulus points, their distances from these are inversely proportional to their shares in the total stimulus.

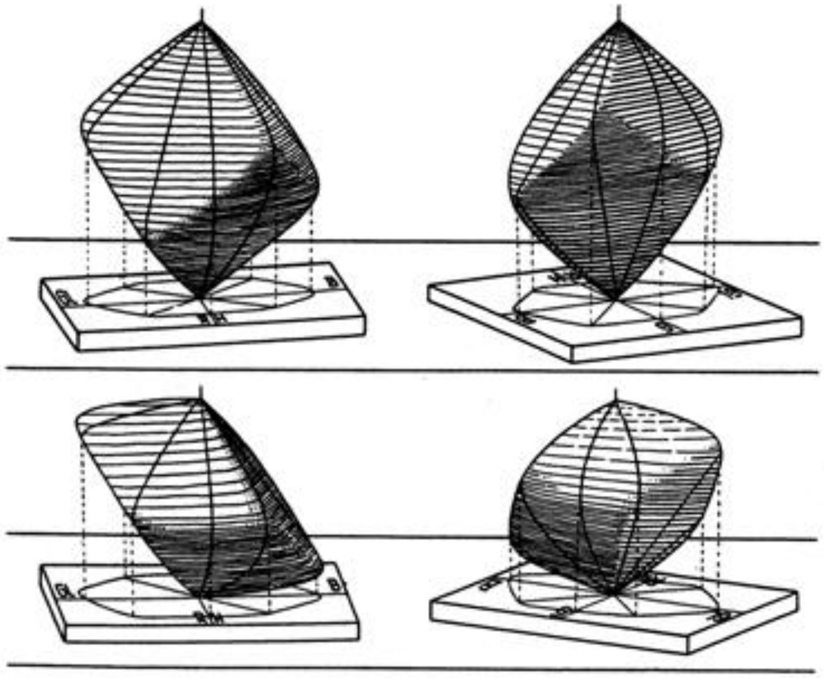


Fig. 12. Perspective representation of the "optimal object color solid," similar to a parallelepiped, with the horizontal coordinates Blueness-Yellowness, Greenness-R(H)edness and the vertical coordinate brightness (top row), stimulus sum (upper row). The meridians of the lower row, with angular difference of approximately 45º, correspond to the sections in the following Fig. 13. Ostwald's full colors are projected onto the horizontal coordinate plane by the mantel of a vertical cylinder touching the surface of the solid.

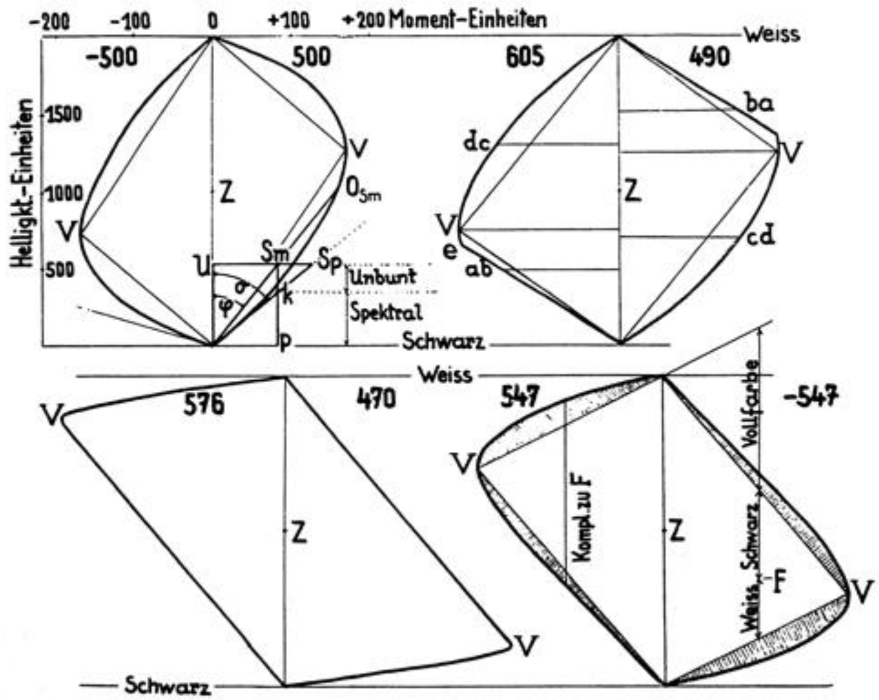


Fig. 13. Four axial sections, corresponding to meridians separated by 45º, through the optimal object color solid of Fig. 12 bottom in brightness units with a common black-white axis and common dual symmetry center *Z*. The sections correspond to the hues Green_{500} - R(H)ed_{500} (top left); Orange_{605} - Green-Blue_{490} (top right); Yellow_{576} – Blue_{470} (bottom left) and Yellow-Green_{547} – $\text{Purple-Violet}_{-547}$ (bottom right). Determination (top left) of saturation $\frac{U-Sm}{U-Sp}=\frac{p-k}{p-Sm}$ and the amount of spectral stimulus $p-k$ in the total brightness $p-Sm$ of the emerald stimulus. Determination of the emerald stimulus *Sm* by darkening of its optimal color O_{Sm}. Representation (top right) of the optimal object colors shown in Fig. 9 with full lines. Representation (bottom right) of Ostwald's triad (hue, and two of the components white, black, full color) and of the colors missing in Ostwald's system (hatched areas). Representation (bottom right) of the totality of the stimuli complementary to stimulus *F*. Exceptional status of yellow object colors.
[Weiss: white, Vollfarbe: full color, Kompl. zu F: complement to F]

16. Every axial section contains the totality of all object color stimuli of two complementary hues. Figure 13 shows four such sections through the solid (in brightness units). In all four sections the mid-point of the black-white axis *Z* is the binomial center of symmetry because two optimal colors that together fill the complete spectrum have identical but opposite color moments, thereby being symmetrical in regard to the center point.

The null point of the coordinate count, – the "black point," – corresponds to infinitely narrow wavelength ranges. Widening of the section results in increases in color moment and brightness in the same sense (see Orange and Green-Blue sections) until the color moment reaches a maximum at the point of Ostwald's full colors (*V*). Further widening results in a complete spectrum being permitted to pass – at the white point –

with the moment again being zero. Each point of the curve can be defined by the two wavelengths that limit the wavelength range. Because of the symmetry of the two complementary halves of the section, a single wavelength scale would suffice. Lines of identical wavelength form on the surface of the solid irregular spirals between the two poles. These, as well as the wavelengths, are not shown in the figures.

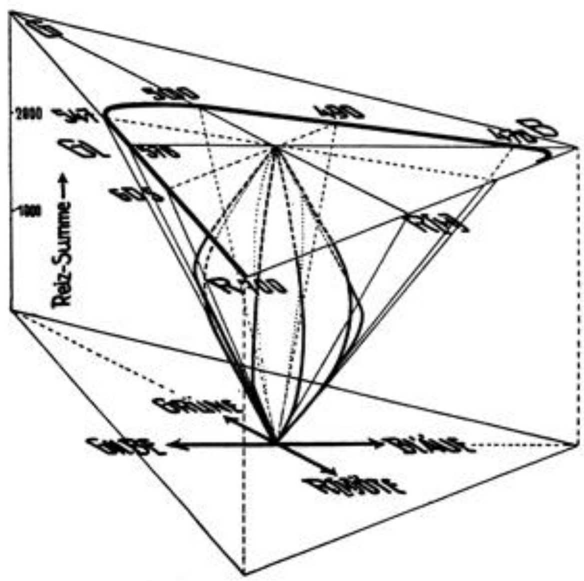


Fig. 14. Relationship between object color stimulus solid, color diagram, and the Maxwell color cone (all in stimulus units).
[Reizsumme: stimulus sum]

The relationship between this – tetrachromatic – optimal object color stimulus solid and the monochromatic and trichromatic systems is evident from the formulas in paragraph 12. They indicate that, for example for the emerald stimulus *Sm* in the Green-R(H)ed section with λ=500 and λ= -500, the

$$\text{length of the lever in the color diagram} = \frac{Moment\ = U - Sm}{Weight\ = p - Sm} = q \text{ tg } \psi.$$

For the saturated spectral color, obtained when the wavelength range is infinitely small, the length of the lever is, therefore, equal to q tg σ. The angle σ is the angle between the tangent at the black point and achromatic axis; the factor q depends on the units of measurement in use and, in the present case, has a value of 0.2. As a result,

$$\text{Saturation} = \frac{tg\psi}{tg\sigma} = \frac{U - Sm}{U - Sp} = \frac{p - k}{p - Sm}.$$

A conclusion from this is that the total brightness of the stimulus *Sm* is composed of the brightnesses $p - k$ of the spectral portion and $k - Sm$ of the achromatic portion.

How the solid relates to the color diagram can also be derived form these formulas and is shown for the stimulus sum solid in perspectival form in Fig. 14. The rays drawn from the black point to the spectral curve (including the purple line) are tangential to optimal object color stimulus solid in the black point and form Maxwell's spectral cone mantle within which all color stimuli are located. Among these the optimal object color stimulus solid delineates all possible object color stimuli.

17. The location of a point in the solid can be, as already partially discussed, indicated in several different ways by three numbers. A specific parameter triad needs to be highlighted because, as it seems, it has become popular in recent times. I have in mind Ostwald's triad in which the three components are, on the one hand, hue, on the other two of the three values that add up to the unity of black, white and full color. It is a fact that within the triangles of white point, black point, and full color each point can be defined from two of the three positive components, in the simplest way (see paragraph 5) in the manner shown for stimulus *F* in Fig. 13, in the section λ = -547. In addition, one notices immediately – see the dashed line regions in this section –, that Ostwald's system clearly does not contain all object colors with positive components. Even though in case of the yellow-blue section 576, 470 the curves and Ostwald's triangles are nearly identical, in case of dark red and light blue-green optimal colors the differences are substantial. Even the emerald stimulus has a negative whiteness component of -0.04, with components of black of +0.57 and of full color of +0.47.

However, it seems to me to be important that for Ostwald's system a simple method for translation into the triad systems of classical color science can actually be found.

18. Figure 13 requires a few additional comments. Stimulus *a b* in the orange section λ = 605 can not only be produced as optimal object color with an appropriately selected wavelength range, but also by darkening of any optimal object color located between points *a b* and *e*. This kind of "metamerism" of optimal colors occurs in explicit form only in the straight part of the spectral curve in the color diagrams, that is, between λ = 700 nm and λ = 575.5 nm. The kink at *e* occurs when one of the two limiting wavelengths of the section passes through the ends of the spectrum.

In the section yellow-green 547 the totality of all stimuli complementary to stimulus F are shown as a vertical straight line.

The especially high saturation (approximately 0.99) and brightness (approximately 0.9 of the total spectrum) of yellow full colors in the neighborhood of λ = 575.5 nm can also be observed in Fig. 13.

Much more could be said concerning the optimal object color stimulus solid, for example how it relates to Ostwalds double cone, Runge's sphere, and the psychological ordering systems of Hering, Ebbinghaus and Höfler, but this would lead too far.[27]

19. I want to conclude with a brief discussion concerning a three-dimensional representation of the totality of all optimal object color stimuli in coordinates identical to those of the optimal object color stimulus solid.

In Fig. 9 the brightnesses of the wavelength ranges from λ = 700 nm to λ are vertically plotted above the points of the moment sum curve that correspond to wavelengths λ. The end points form a three-dimensional screw line showing two revolutions of the screw (see Fig. 15).

The totality of all chords pointing upwards, that is, the straight lines connecting two points on a turn of the screw, represents the totality of all optimal object colors. The non-optimal stimuli, generated by weakening or darkening of their corresponding optimal stimuli, are represented by sections of the chords.

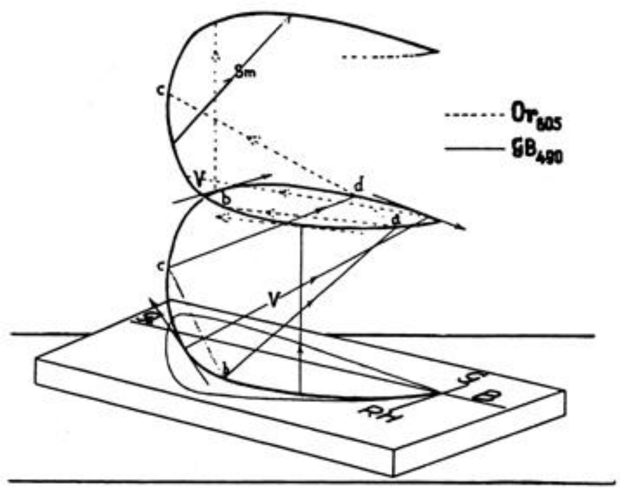


Fig. 15. Perspectival representation of the "optimal object color spiral." Color stimuli are represented by vectors with the components: Blueness-Yellowness, Greenness-R(H)edness, and brightness. The plotted straight lines and letters correspond to those in Figs. 9, 10, and 13. Fig. 9 is a vertical parallel projection of the three-dimensional structure of Fig. 15.

In other words, for each optimal object color stimulus there is a certain corresponding vector, the three vertical components of which are the three independent parameters: brightness (vertical toward the top) and the two primary stimulus moments (horizontal). The additional parameters, hue, lengths of the lever in the color diagram, saturation, brightness of the spectral portion, etc., are derived in the same way as in case of the earlier discussed color stimulus solid. The vectors of this "optimal object color spiral" used to represent the stimuli – assuming identical scales – are identical to the vectors originating in the black point (paragraph 15) of the color stimulus solid of Fig. 12 (lower row) and Fig. 13, and are only arranged in a different spatial manner. They again are lines of identical color character, that is, identical stimulus kind (equal hue and equal saturation). They correspond to Ostwald's "shadow series" and are related to the horizontal lines of Fig. 10. The stimuli identified in Fig. 15 are the same, carrying the same designations, as those in Fig. 9, Fig. 13 upper row, and Fig.10 left. The moment sum curve of Fig. 9, containing cords, partial chords, and tangents, is a vertical parallel projection of the "optimal object color spiral of Fig. 15.

Also in case of this spatial structure I have to limit myself to brief comments. In the second part of my presentation I will briefly touch on a few subjects of color measurement technology and the application of the science of color stimuli.

20. Several different kinds of methods of color measurement and heterochromatic photometry have been proposed that, by the way, can in my opinion be clearly ordered into a heuristically valid system. Here I only concern myself with objective methods because they are directly related to a few practically important problems, to be discussed briefly below.

In a certain sense the calculation of the three primary stimuli from the experimentally determined emission, transmission or reflectance spectra, discussed in paragraphs 3 and 4, is already an objective method because the determination of these spectra is independent of the specific properties of the measuring eye and can be achieved by means of objective radiation sensors, for example a thermopile. The necessary summation of the primary stimuli of the individual, narrow spectral regions over the total spectrum can now also be done directly by the instrument itself, as shown by H. Ives.[28]

According to Ives' procedure (Fig. 16), a spectrum *Sp* is unified with a collecting lense *L*, today frequently employed, for example also in the large Zeiss apparatus, and going back to Pouillet and Foucault. The result is a uniformly colored image of a diaphragm *Bl* located near the prism. This image falls onto a suitable summing radiation detector *E*, a [thermosaeule], a bolometer, a photoelectric cell, or similar equipment, connected to an indicator instrument that also sums – such as a galvanometer. Differently formed masks *Sch* can be applied to the spectrum whereby its height, and as a result the energy transmission, in the specific wavelength regions λ, can be reduced in the ratio Sch_λ :1, assuming that the diaphragm is uniformly illuminated and that the spectrum is sufficiently free of light from other sources.

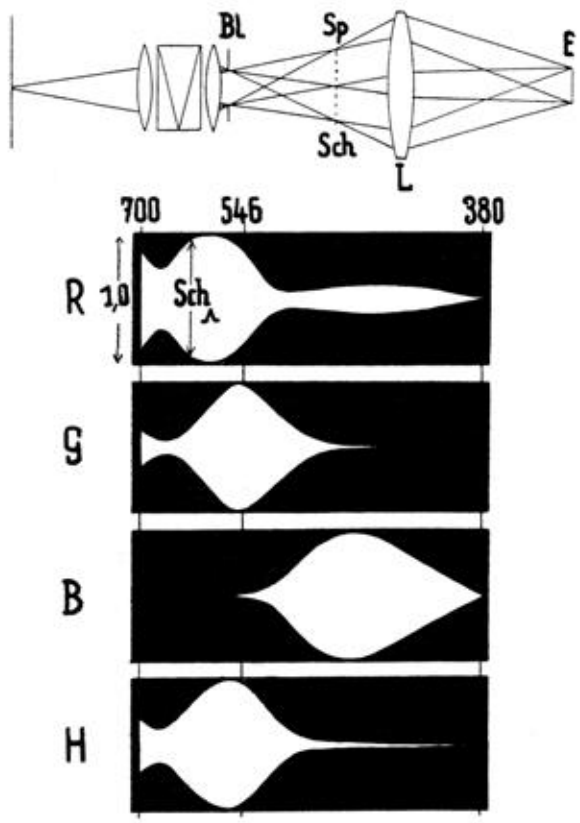


Fig. 16. Light path and spectral masks for objective trichromatic color analysis and objective heterochromatic photometry according to H. Ives.

E_λ is defined as the value indicated by the instrument as the result of daylight spectrum irradiation of the sensor by the full strength section of width $d\lambda$ at wavelength λ. E_λ can therefore be described as the spectral daylight sensitivity of the sensor in the spectral range λ. After inserting the mask the instrument indicates a value A_{Sch} *(λ)* equal to $E_\lambda \cdot Sch_\lambda \cdot d_\lambda$. If we now insert into the light beam a colored film *F*, the transparent color and lightness of which is to be measured, and designate its transmittance at wavelength λ as F_λ, the resulting value is A_{Sch+F} *(λ)* equal to $E_\lambda \cdot Sch_\lambda \cdot F_\lambda \cdot d_\lambda$. The sum values with inserted mask and mask + colored film for the total visible daylight spectrum, obtained by summation, or integration of the individual results at each wavelength from λ = 700 nm to λ = 380 nm, are

$$\mathcal{A}_{\mathrm{Sch}} = \int A_{Sch}(\lambda) = \int E_\lambda \cdot Sch_\lambda \cdot d\lambda \quad \text{and} \quad \mathcal{A}_{\mathrm{Sch+F}} = \int A_{Sch+F}(\lambda) = \int E_\lambda \cdot Sch_\lambda \cdot F_\lambda \cdot d\lambda .$$

In comparison, the formulas (paragraph 3) for the amounts of red basic stimulus without ($\mathcal{R}_{\mathrm{W}}$) and with ($\mathcal{R}_{\mathrm{W+F}}$) of the colored film to be measured are

$$\mathcal{R}_{\mathrm{W}} = \int R_\lambda \cdot d\lambda \quad \text{and} \quad \mathcal{R}_{\mathrm{W+F}} = \int R_\lambda \cdot F_\lambda \cdot d\lambda .$$

Therefore, if the mask is cut in such a manner that $E_\lambda \cdot Sch_\lambda = R_\lambda$, comparison of the two pairs of formulas indicates that the measured result $\mathcal{A}_{\mathrm{Sch+F}}$ is proportional to the red basic stimulus value $\mathcal{R}_{\mathrm{W+F}}$. The proportionality factor is $\mathcal{A}_{\mathrm{Sch}} / \mathcal{R}_{\mathrm{W}}$. The same idea can be applied to the other two basic color stimuli and to brightness, so that an objective, that is, valid for the "average eye," measurement of color and a corresponding heterochromatic photometry is possible.

Figure 16 shows the masks used by Ives to represent his three basic color stimuli and brightness. The similarity with the curves of Fig. 1 (and their mirror image) is apparent. Obvious deviations are due to special properties of the prism (dispersion, absorption) and the light source (artificial light with daylight filter). The validity of the objective measuring technique was experimentally demonstrated by Ives.

In place of a spectrum weakened by masks it is possible to use undispersed light with suitable selective filters. In that case it is of course necessary that the spectral transmittance T_λ of the filter, for example for the red stimulus determination, is in agreement with the following precondition: $T_\lambda \cdot E_\lambda = R_\lambda$. Several such selective filters have been proposed for the purpose of objective heterochromatic photometry with "gray" sensors,[29] and used successfully.[30] However, selective filters have as yet not been proposed for the purpose of objective trichromatic color analysis.

21. But L. Bloch has pursued this path in his subjective optical method of color measurement, using trichromatic filter analysis.[31] The theory behind his, in principle, valid method can be easily comprehended based on what has just been discussed. In all formulas the spectral sensitivity of the sensor E_λ has to be replaced by the spectral sensitivity of the normal daylight eye H_λ (see Fig. 5). In other words, transmittance T_λ of the selective filters, for example for the red basic color stimulus, must relate to

wavelength in such a manner that the expression $\frac{T_\lambda \cdot H_\lambda}{R_\lambda}$ is constant across the complete spectrum.

Calculations for the three primary stimuli I selected result in partially overlapping[32] transmittance curves with transmittance as high as possible, shown in Fig. 17. The brightness of the red filter is approximately 0.46, of the green filter 0.74, and of the blue filter 0.04 of the brightness of unfiltered daylight. The very low brightness of the light passing through the blue filter is a disadvantage of this otherwise very convenient method. This negative aspect can be somewhat mitigated by selecting different primary stimuli. But such a change results in reduced accuracy. It should be noted here that in many practical situations the transmittance curves of the three filters do not need to meet the theoretical requirements accurately: the greyer the light source impacting on the color to be measured, the more uniform the reflectance or transmittance curves of the colors to be measured are, the larger the tolerable deviations from theory. The colored glass filters used by Bloch in his trichromatic analysis, the transmittance curves of which I gratefully received in a letter from Dr. Bloch, cannot be brought into agreement with any of the known spectrum standardizations. Nevertheless, color measurements of heat radiators that Bloch made according to his method agreed very closely with values obtained by Ives according to a totally different additive method; however, in case of the light from a mercury arc lamp there are, as to be expected, significant deviations.

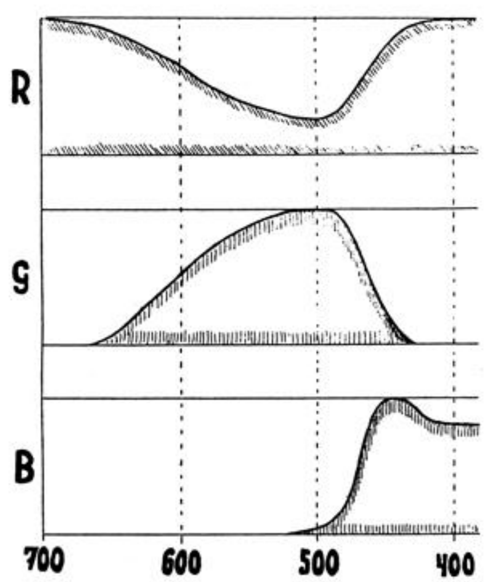
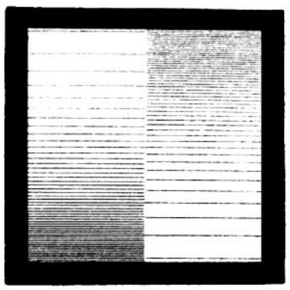


Fig. 17. Theoretical filters of highest brightness for subjective trichromatic color analysis according to L. Bloch.

22. At this point I would like to make a brief excursion, related to the technology of heterochromatic photometry, for example also with essentially monochromatic filters according to Fig. 17. It is generally known that determining with certainty the identity of or difference in brightness of two adjacent parts of a photometric field with different colors is very difficult. This is immediately improved by shading, according to a in my view insufficiently considered proposal by Pfund[33], the two halves of a symmetric

photometric field with a neutral gray wedge, as schematically shown in Fig. 18: there appears a slanted figure 8. From its degree of symmetry it is relatively easy to determine if equality of brightness is achieved.

In a general sense related to this is the flicker-photometric procedure introduced by Pfund (see note 33). If two equal-sized heterochromatic fields shaded in opposite direction – such as the two sides of Fig. 18 – are made to flicker together, flickering occurs on top and bottom because of differences in brightness; in between there is a zone free of flicker, the width of which can be reduced to a minimum by adjusting the flicker frequency. The location of the flicker-free zone can, even by people with little practical experience, be adjusted with a high degree of certainty to fall into the center of the visual field by adjusting the two gray wedges.[34]



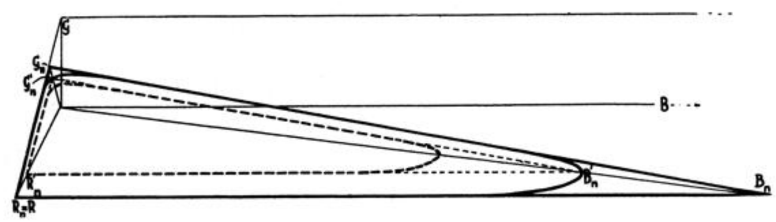


Fig. 18. Photometric field according to Pfund for heterochromatic photometry (schematical).

Fig. 19. Selection of projection or viewing filters R_n', G_n', B_n' for reproducing all simple and complex color stimuli accurate in hue and brightness, but with constant reduced saturation (≈ ¾ of the original).

23. After this excursion into measuring technology I am now returning to a practically especially important case of objective trichromatic filter analysis and synthesis: three-color photography. Here the negative plate in a sense represents the role of the "sensor", the positive image that of "indicating instrument."

Let us first look at the most simple case: additive synthesis using three achromatic positive slides through each of which differently colored light passes, be it with the help of projecting one on top of the other, or by viewing them in a chromoscope. The three positive slides have to fulfill the condition that their transmittance at each point is proportional to the amount of light that impacted on the corresponding points of their negatives. It is possible to achieve this with good approximation.[35]

Consideration of the color solid (Figs. 12 and 13) results in the totality of all object colors to be given account of; at the same time it indicates that, with the help of three quite accurately determinable optimal object colors nearly all of the remaining object color stimuli can be produced by means of vectorial addition. Therefore, in additive color measurement procedures these three stimuli, with hues corresponding approximately to wavelengths 620, 540, and 455 nm, will need to be employed.

These three stimuli, however, are not automatically useful in three-color photography: spectral and object color stimuli are not reproduced exactly in regard to brightness and hue or with a saturation independent of the hue, or they are – when hue and brightness are to be reproduced accurately and saturation for all stimuli reduced in a constant ratio, they will be reproduced with very low saturation. If the latter requirement (hue and brightness correct, saturation in constant ratio to the saturation of the original stimuli) is to be achieved at the highest possible saturation, it is necessary, as simple consideration indicates, to resort to additive trichromatic projection or viewing of three stimuli that correspond to the corners of the inner triangle in Fig. 19. Every narrow section of the (solid line) spectral curve of the original is represented by the corresponding section of the less saturated inner (dashed) curve. So that the saturation remains the same across the whole spectrum and thereby also in all mixed stimuli, and as high as possible, a very specific selection of the stimuli G_n and B_n must be made, as indicated in Fig. 19. As a result, and expressed in brightness units, the saturation will be everywhere approximately three quarters of that of the original.

When calculating, for a given panchromatic emulsion, the transmittance curves of the photo filters necessary under these conditions, the results shown in Fig. 20 are obtained. What is characteristic in these curves is the broad overlaps, necessary if also narrow spectral ranges (spectra) are to be reproduced in colors similar to the original. If there are only broad spectral ranges (object colors) in the original, significantly different projection and exposure filters can be selected on basis of multiple special conditions and compromises. However, in that case defined rules can no longer be established.

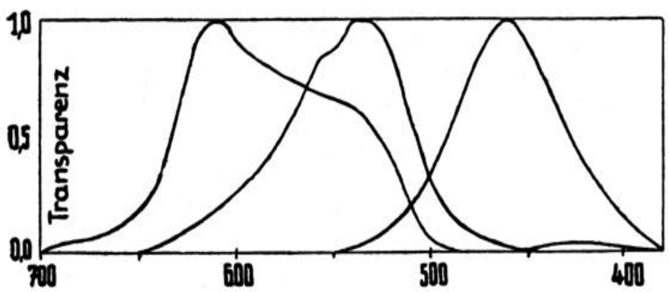


Fig. 20. Spectral transmittance of photographic filters for the additive synthesis according to Fig. 19 (for a specific panchromatic emulsion).
[Transparenz: transmittance]

Theoretically even more complicated, and thereby under practical conditions requiring even more compromises, are additive processes using halftone printing plates,[36] where usually the filters for synthesis and analysis are the same, or subtractive methods, such as Pinatypie or Uvachromie,[A] etc., and finally the technically very important mixed subtractive-additive processes of reproduction technology.

24. In all such compromises a psychologically ordered color system, a system with colors separated according to "equidistant" steps in Hering's sense or according to just noticeable differences, would be of much importance and help avoid much arbitrariness. Schrödinger, with his higher color metric, has produced a general mathematical theory, quantitative scaling attempts have been made by Ebbinghaus, Höfler, Ostwald, and others, there is already a considerable amount of experimental data in regard to distinction thresholds for brightness, saturation, and hue, but many more experimental and theoretical efforts will be required before such a system of psychological order can be considered as more or less complete and can be seen as approximately equivalent to the color stimulus system.

25. Complicating on one side for the corresponding experiments, on the other side possibly simplifying practical application are the contributory factors of psychological phenomena, such as simultaneous and successive contrast, change in apparent color as a result of illumination and adaptation, the connection by experience of color on the one hand and form, distance, size on the other – all phenomena that influence our color judgments to a surprisingly large degree.

Those who have only little experience in this field, those that have seen experiments demonstrating color contrast, going back to Goethe and Chevreul, those who have repeated the astonishing experiments of Hering and Katz concerning the manifestations of color or have contemplated the results of experiments on the influence of hypnosis on color contrast, those that of an evening, based on the white appearance of writing paper illuminated with the light of a lamp, have marveled at the huge scope of our subjective color perception and have recognized purely subjective filters in the artificial light filters of halftone images, those who have compared a colored image printed on paper with a colored – additively or subtractively generated – projected image and the colored original itself, those who have subjected themselves to the apparently truthful, but objectively wrongly colored projection images, those who have viewed the colored surroundings upside down on a camera screen, or viewing them from between the legs ["beinlings"], as mentioned by Helmholtz,[37,B] those who have contemplated the percepts of black, white, and brown, those who when viewing a high quality two-color photo image have been surprised by experiencing colors objectively not present, all those will recognize the normal impact of central psychological processes on our position relative to the diversity of colors and will no doubt agree with me that the well-known remark by Karaffa "Mundus vult decipi, ergo decipiatur" [C] [The world wants to be deceived, so let it be deceived] applies hardly anywhere else better than to the field of applied color science, if the word "mundus" [world] is replaced by "oculus" [eye].

As a result, I see the systematization of this "deceptio oculi," this deception of the eye, to be a very important task of practically applied color science. I believe that this already frequently attacked field of color perception, despite the complexity of its phenomena, some day can be forced into a closed system that might be more important for the technologist than for the pure scientist.

Conclusion

There are an infinite number of parameter triads that equally completely represent the properties of color stimuli.

In regard to the classical three-primary-stimuli triad the trick has been mentioned that the determination of the three primary stimulus amounts in the spectral curve of a mixed stimulus can be simplified with the help of spectra arranged to be "equally red", "equally green", and "equally blue."

The magnitude "color moment" has been derived from Newton's center of gravity point of view for application in the matter of summation of color stimuli. It can be useful for the illustration of the relationship between the "three-color system" and the "four-color system" of color stimuli emphasized by Pilgrim and Schrödinger.

When broadening the wavelength range without changing the hue the two limiting wavelengths of the section can be easily determined by graphical methods with the aid of the “spectral moment sum curve”.

In addition, it can be determined that Ostwald’s “full colors” also represent a special case in the sense of classical color stimulus science: they contain the highest amount of spectral stimulus that is possible among object colors of equal hue.

In particular, three three-dimensional color solids important for object color stimuli can be constructed: a) a (Grassmann-Helmholtz) solid (with the coordinates hue, saturation, brightness) that also makes it possible to read the wavelength range of “optimal object colors,” b) an “optimal object color spiral” (with the coordinates Blueness-Yellowness, Greenness-R(H)edness, brightness) that also makes it possible to read the wavelength ranges of “optimal object colors” and, finally c) a (Hering-Maxwell) “optimal object color solid” with the same coordinates.

The latter construct directly demonstrates the relationship between Ostwald’s parameter triad and the classical color stimulus theory, as well the object color stimuli missing in Ostwald’s system.

In connection with the objective method of color and brightness measurement introduced by Ives, the theory of Bloch’s method of color measurement using subjective trichromatic filter analysis has been discussed.

Using additive three-color photography all complex and simple color stimuli can be reproduced accurately in hue and brightness, but with a constant relative diminution of saturation compared to the original.

The reader has been reminded of the high practical importance of psychological phenomena when judging color perceptions.

(Submitted Sep. 20, 1927)

Notes:

1. Ann. d. Physik **63** (1920).
2. Especially by E. Schrödinger in Müller-Pouillet’s *Lehrbuch der Physik* (1926).
3. Zeitschr. f. Psychol. u. Physiol. d. Sinnesorgane, Supplementary volumeVII.
4. Paragraphs 3 up to and including 19 were to be published with essentially identical content, but slightly differing in the presentation from the current version, in Sept. 1923 as a contribution to the Ostwald-Celebration edition of Zeitschr. f. angew. Chemie. Because of the high printing cost (inflationary time period!) of the numerous figures the article could not be printed and at the time only the title “Spectrum and Color” was published. See Zeitschr. f. angew. Chem. **36** (1923).
5. I constructed the curves from measurements by König and Dieterici, v. Kries, Maxwell, Abney, Exner, Angier and Trendelenburg, among others, not without a degree of unavoidable arbitrariness when converting the various data sets. The numbers will be published elsewhere. A sufficiently close agreement was found upon subsequent recalculation for the primary stimuli selected by Ives (Frankl.

Inst. 1915 and 1923). But the primary stimuli I selected have a peculiarity to be discussed below (end of paragraph 7).
I take the opportunity to express my most sincere thanks to W. Bartsch and doctoral candidate M. Wend for their help with the figures, the latter in addition for reviewing numerical calculations and cooperation in producing the models.

6. Wien. Akad. Ber. **III** (1902).
7. The areas are most easily determined by weighing them against counter weights made from the same kind of paper divided into small squares.
8. The wavelengths of the resulting hue and the peak of reflection do not necessarily coincide.
9. In the Anglo-American literature this triad is known as the "monochromatic system." For "saturation" (in brightness units) the expression "purity" is frequently used.
10. Hue and saturation or two components of primary stimuli (without indication of the amount or intensity) define *Reizart* (v. Kries) or the "color character" or the "chroma" of a color stimulus.
11. Philos. Mag. **24** (1912).
12. The only method with which the results were strongly deviating was the sharpness-of-vision method (because of the chromacy of the eye to which, for us wearers of glasses, the chromacy of the glasses is added.
13. Compare especially also Cl. Schäfer, Physikal. Zeitschr. **26** (1925).
14. A. Mitscherling (Wundt's Psychol. Studien I, Issue 2 (1905) as well as Angier and Trendelenburg (Zeitschr. f. Psychol. u. Physiol. d. Sinnesorg. 39 (1905) were, to my knowledge, the first making direct, experimental utilization of the color chart in brightness units.
15. If in a different context, I find this curve, with the designation "spectral color curve," first shown by L. Pilgrim: Einige Aufgaben der Wellen- und Farbenlehre des Lichtes (Beilage zum Programm der Realanstalt Cannstadt **1901**; see also Lehmann-Frick II, 2, p. 1793 **1901**).
16. Two of the three curves having in the color diagrams of Fig. 5 a wavelength scale represent with equal accuracy the complete content of classical color theory, as do three of the four curves in the corresponding lower figures.
17. The dashed parts of the right color diagram are slanted central projections of the corresponding curves in the left diagram.
18. H. Ives, Frankl. Inst. 1915. The spectral curve in brightness units nevertheless remains always finite and on the right of the *R-G* line, because the experimentally determined spectral brightness curve is independent of the arbitrarily selected primary stimuli.
19. See E. Schrödinger, Wien. Akad. Ber. **134** (1925). For practical calculations it is very convenient that in this case the two other coefficients λ and ρ are so close to identical that, for practical calculations they can be set as $\lambda = \rho$. As a consequence, for the primary stimulus triad I selected, the conversion from stimulus units to brightnesses is much simplified: the brightness of any particular stimulus is simply the sum of the red stimulus + the green stimulus if $\lambda = \rho = 1$, that is, if the brightness of the total spectrum = 2000 brightness units.

20. Compare with paragraphs 8 and 9 the related, but not identical expositions by Pilgrim, loc. cit., and by Schrödinger, Wien. Akad. Ber. **134** (1925), see also Note 4.
21 One might perhaps call it "tetrachromatic system."
22. Coefficients *m, n, p* depend on the selection of the axes, the color diagram, and the units of measurement. If the axes cross the achromatic point the following relationship applies: $m + n + p = 0$. In my case

$$M_{\text{Blue-Yellow}} = 1/3\, B - 1/3\, R; \quad M_{\text{Green-R(H)ed}} = 2/3\, G - 2/3\, R;$$

$$S = R + G + B; \quad H = R + G.$$

23. Again (see paragraph 4) one can apply the trick by placing the spectrum on the abscissa axis in such a manner that equal increments of the abscissa correspond to equal increments in the primary stimulus, the stimulus sum, or brightness. The daylight spectrum of constant luminosity constructed by this method shall, from here on, be named "equiluminous spectrum." To my best knowledge, it was introduced by A. Fick, Pflügers Arch. **64** (1896).
24. E. Schrödinger, Ann. d. Physik **62** (1920). W. Ostwald, Physikal. Zeitschr. **17** (1916). On the light fastness of corresponding pigments: R. Luther, Zeitschr. f. Elektrochem. **14** (1908).
25. I owe first knowledge of this parameter triad to discussions with Prof. Dr. E. Goldberg before the war. The corresponding article, part of a series of articles, Zeitschr. f. Reproduktionstechnik 1914, was not published due to the beginning of the war. The same triad was later recommended by v. Kries, This journal **5** (1924).
26. Photographic Journal 37 (1913). The idea was derived during some earlier conversation with me. At the time (1906) I did not have knowledge of the spectral moment sum curve.
27. There is also a certain resemblance with Munsell's "color tree," so far known to me only from presentations of papers. However, according to what I have learned, his three parameters are hue, brightness, and saturation. Accordingly, axial sections should stand in a simple relationship to the curves of Fig. 10.
28. Physic. Rev. 6 (1915). Artificial precision eye.
29. See also H. Schulz, *Das Sehen*, p. 50, Verlag Enke 1920.
30. H. Ives and E. Kingsbury, Physic. Rev. **6** (1915).
31. Elektrotechn. Zeitschr. 1913.
32. See however, A. von Hübl, Physikal. Zeitschr. **18** (1917).
33. Phys. Rev. **4** (1914). Zeitschr. f. Instrum. **35** (1915).
34. This phenomenon can also be demonstrated very effectively in a larger auditorium with the help of a double projection apparatus with a rotating sector diaphragm at the location of the light source images.
35. To achieve this, the product of the tangents of inclination of the "characteristic curves" of the related negatives and slides for projection requires always a value of approximately 0.6, for viewing in a chromoscope a value of 1.
36. See also Cl. Schaefer and K. Ackermann, This journal **8** (1927).

37. Hans Hoffmann's humorous poem with the same title and beginning as follows:

I met a strange man
Who watched the world through his legs ["beinlings"]
That is by hanging his head
Through the legs his own body grew …

may possibly refer to Helmholtz.

A. Pinatypie and Uvachromie: two photographic three-color reproductive processes of the early 20th century. Pinatypie, invented in 1903 by Léon Didier and developed into a commercial process by chemists at Hoechst AG in Germany, was based on bichromate/gelatin plates. The Uvachromie process was invented by Arthur Traube and first described in 1916. Traube founded Uvachrom AG in 1918, a company active until the 2nd World War.

B. Note 37: It is not evident to what related quotation from Helmholtz the author refers.

C. It is not evident to which Karaffa Luther attributes the quoted phrase. It is generally attributed to the first century CE Roman poet Petronius.